

THE HARMONIES OF THE WORLD

by Johannes Kepler

Concerning the very perfect harmony of the celestial movements, and the genesis of eccentricities and the semidiameters, and the periodic times from the same.

After the model of the most correct astronomical doctrine of today, and the hypothesis not only of Copernicus but also of Tycho Brahe, whereof either hypotheses are today publicly accepted as most true, and the Ptolemaic as outmoded.

I commence a sacred discourse, a most true hymn to God the Founder, and I judge it to be piety, not to sacrifice many hecatombs of bulls to Him and to burn incense of innumerable perfumes and cassia, but first to learn myself, and afterwards to teach others too, how great He is in wisdom, how great in power, and of what sort in goodness. For to wish to adorn in every way possible the things that should receive adornment and to envy no thing its goods—this I put down as the sign of the greatest goodness, and in this respect I praise Him as good that in the heights of His wisdom He finds everything whereby each thing may be adorned to the utmost and that He can do by his unconquerable power all that he has decreed.

GALEN, *on the Use of Parts*. Book III

PROEM

[268] As regards that which I prophesied two and twenty years ago (especially that the five regular solids are found between the celestial spheres), as regards that of which I was firmly persuaded in my own mind before I had seen Ptolemy's *Harmonies*, as regards that which I promised my friends in the title of this fifth book before I was sure of the thing itself, that which, sixteen years ago, in a published statement, I insisted must be investigated, for the sake of which I spent the best part of my life in astronomical speculations, visited Tycho Brahe, [269] and took up residence at Prague: finally, as God the Best and Greatest, Who had inspired my mind and aroused my great desire, prolonged my life and strength of mind and furnished the other means through the liberality of the two Emperors and the nobles of this province of Austria-on-the-Anisana: after I had discharged my astronomical duties as much as sufficed, finally, I say, I brought it to light and found it to be truer than I had even hoped, and I discovered among the celestial movements the full nature of harmony, in its due measure, together with all its parts unfolded in Book III—not in that mode wherein I had conceived it in my mind (this is not last in my joy) but in a very different mode which is also very excellent and very perfect. There took place in this intervening time, wherein the very laborious reconstruction of the movements held me in suspense, an extraordinary

augmentation of my desire and incentive for the job, a reading of the *Harmonies* of Ptolemy, which had

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been sent to me in manuscript by John George Herward, Chancellor of Bavaria, a very distinguished man and of a nature to advance philosophy and every type of learning. There, beyond my expectations and with the greatest wonder, I found approximately the whole third book given over to the same consideration of celestial harmony, fifteen hundred years ago. But indeed astronomy was far from being of age as yet; and Ptolemy, in an unfortunate attempt, could make others subject to despair, as being one who, like Scipio in Cicero, seemed to have recited a pleasant Pythagorean dream rather than to have aided philosophy. But both the crudeness of the ancient philosophy and this exact agreement in our meditations, down to the last hair, over an interval of fifteen centuries, greatly strengthened me in getting on with the job. For what need is there of many men? The very nature of things, in order to reveal herself to mankind, was at work in the different interpreters of different ages, and was the finger of God—to use the Hebrew expression; and here, in the minds of two men, who had wholly given themselves up to the contemplation of nature, there was the same conception as to the configuration of the world, although neither had been the other's guide in taking this route. But now since the first light eight months ago, since broad day three months ago, and since the sun of my wonderful speculation has shone fully a very few days ago: nothing holds me back. I am free to give myself up to the sacred madness, I am free to taunt mortals with the frank confession that I am stealing the golden vessels of the Egyptians, in order to build of them a temple for my God, far from the territory of Egypt. If you pardon me, I shall rejoice; if you are enraged, I shall bear up. The die is cast, and I am writing the book—whether to be read by my contemporaries or by posterity matters not. Let it await its reader for a hundred years, if God Himself has been ready for His contemplator for six thousand years.

The chapters of this book are as follows:

1. Concerning the five regular solid figures.
2. On the kinship between them and the harmonic ratios.
3. Summary of astronomical doctrine necessary for speculation into the celestial harmonies.
4. In what things pertaining to the planetary movements the simple consonances have been expressed and that all those consonances which are present in song are found in the heavens.

5. That the clefs of the musical scale, or pitches of the system, and the genera of consonances, the major and the minor, are expressed in certain movements.
6. That the single musical Tones or Modes are somehow expressed by the single planets.
7. That the counterpoints or universal harmonies of all the planets can exist and be different from one another.
8. That four kinds of voice are expressed in the planets: soprano, contralto, tenor, and bass.
- [270] 9. Demonstration that in order to secure this harmonic arrangement, those very planetary eccentricities which any planet has as its own, and no others, had to be set up.
10. Epilogue concerning the sun, by way of very fertile conjectures.

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Before taking up these questions, it is my wish to impress upon my readers the very exhortation of Timaeus, a pagan philosopher, who was going to speak on the same things: it should be learned by Christians with the greatest admiration, and shame too, if they do not imitate him: Ἄλλ' ὃ Σὼκράτης, τοῦτο γε δὴ πντες ὅσοι καὶ κατὰ βραχὺ σωφροσύνης μετέχουσιν, ἐπὶ πασῇ ὀρμῇ καὶ σμίκροῦ καὶ μεγάλου πράγατος θεὸν ἀει που καοῦσιν. ἡμᾶς δὲ τοὺς περὶ τοῦ πάντος λόγους ποιεῖσθαι πη μέλλοντας. . . , εἰ μὴ πανταπασι παραλλάττομεν, ἀνάγκη θεοὺς τε καὶ θεᾶς ἐπικαλουμένους εὔχεσθαι πάντα, κατὰ νοῦν ἐκεῖνοις μὲν μάλιστα, ἐπομένως δὲ ἡμῖν εἰπεῖν. *For truly, Socrates, since all who have the least particle of intelligence always invoke God whenever they enter upon any business, whether light or arduous; so too, unless we have clearly strayed away from all sound reason, we who intend to have a discussion concerning the universe must of necessity make our sacred wishes and pray to the Gods and Goddesses with one mind that we may say such things as will please and be acceptable to them in especial and, secondly, to you too.*

1. CONCERNING THE FIVE REGULAR SOLID FIGURES

[271] It has been said in the second book how the regular plane figures are fitted together to form solids; there we spoke of the five regular solids, among others, on account of the plane figures. Nevertheless their number, five, was there demonstrated; and it was added why they were designated by the Platonists as the figures of the world, and to what element any solid was compared on account of

what property. But now, in the anteroom of this book, I must speak again concerning these figures, on their own account, not on account of the planes, as much as suffices for the celestial harmonies; the reader will find the rest in the *Epitome of Astronomy*, Volume II, Book IV.

Accordingly, from the *Mysterium Cosmographicum*, let me here briefly inculcate the order of the five solids in the world, whereof three are primary and two secondary. For the cube (1) is the outmost and the most spacious, because firstborn and having the nature [*rationem*] of a *whole*, in the very form of its generation. There follows the *tetrahedron* (2), as if made a *part*, by cutting up the cube; nevertheless it is primary too, with a solid trilinear angle, like the cube. Within the tetrahedron is the *dodecahedron* (3), the last of primary figures, namely, like a solid composed of parts of a cube and similar parts of a tetrahedron, *i.e.*, of irregular tetrahedrons, wherewith the cube inside is roofed over. Next in order is the *icosahedron* (4) on account of its similarity, the last of the secondary figures and having a plurilinear solid angle. The *octahedron* (6) is inmost, which is similar to the cube and the first of the secondary figures and to which as inscriptile the first place is due, just as the first outside place is due to the cube as circumscriptile.

[272] However, there are as it were two noteworthy weddings of these figures, made from different classes: the males, the cube and the dodecahedron, among the primary; the females, the octahedron and the icosahedron, among the secondary, to which is added one as it were bachelor or hermaphrodite, the tetrahedron, because it is inscribed in itself, just as those female solids are inscribed in the males and are as it were subject to them, and have the signs of the feminine sex, opposite the masculine, namely, angles opposite planes. Moreover, just as the tetrahedron is the element, bowels, and as it were rib of the male

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cube, so the feminine octahedron is the element and part of the tetrahedron in another way; and thus the tetrahedron mediates in this marriage.

The main difference in these wedlocks or family relationships consists in the following: the ratio of the cube is *rational*. For the tetrahedron is one third of the body of the cube, and the octahedron half of the tetrahedron, one sixth of the cube; while the ratio of the dodecahedron's wedding is *irrational* [*ineffabilis*] but *divine*.

The union of these two words commands the reader to be careful as to their significance. For the word *ineffabilis* here



does not of itself denote any nobility, as elsewhere in theology and divine things, but denotes an inferior condition. For in geometry, as was said in the first book, there are many irrationals, which do not on that account participate in a divine proportion too. But you must look in the first book for what the divine ratio, or rather the divine section, is. For in other proportions there are four terms present; and three, in a continued proportion; but the divine requires a single relation of terms outside of that of the proportion itself, namely in such fashion that the two lesser terms, as parts make up the greater term, as a whole. Therefore, as much as is taken away from this wedding of the dodecahedron on account of its employing an irrational proportion, is added to it conversely, because its irrationality approaches the divine. This wedding also comprehends the solid star too, the generation whereof arises from the continuation of five planes of the dodecahedron till they all meet in a single point. See its generation in Book

Lastly, we must note the ratio of the spheres circumscribed around them to those inscribed in them: in the case of the tetrahedron it is rational, 100,000 : 33,333 or 3 : 1; in the wedding of the cube it is irrational, but the radius of the inscribed sphere is rational in square, and is itself the square root of one third the square on the radius [of the circumscribed sphere], namely 100,000 : 57,735; in the wedding of the dodecahedron, clearly irrational, 100,000 : 79,465; in the case of the star, 100,000 : 52,573, half the side of the icosahedron or half the distance between two rays.

2. ON THE KINSHIP BETWEEN THE HARMONIC RATIOS AND THE FIVE REGULAR FIGURES

[273] This kinship [*cognatio*] is various and manifold; but there are four degrees of kinship. For either the sign of kinship is taken from the outward form alone which the figures have, or else ratios which are the same as the harmonic arise in the construction of the side, or result from the figures already constructed, taken simply or together; or, lastly, they are either equal to or approximate the ratios of the spheres of the figure.

In the first degree, the ratios, where the character or greater term is 3, have kinship with the triangular plane of the tetrahedron, octahedron, and icosahedron;

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but where the greater term is 4, with the square plane of the cube; where 5, with the pentagonal plane of the dodecahedron. This similitude on the part of the plane can also be extended to the smaller term of the ratio, so that wherever the number 3 is found as one term of the continued doubles, that ratio is held to be akin to the three

figures first named: for example, 1 : 3 and 2 : 3 and 4 : 3 and 8 : 3, et cetera; but where the number is 5, that ratio is absolutely assigned to the wedding of the dodecahedron : for example, 2 : 5 and 4 : 5 and 8 : 5, and thus 3 : 5 and 3 : 10 and 6 : 5 and 12 : 5 and 24 : 5. The kinship will be less probable if the sum of the terms expresses this similitude, as in 2 : 3 the sum of the terms is equal to 5, as if to say that 2 : 3 is akin to the dodecahedron. The kinship on account of the outward form of the solid angle is similar: the solid angle is trilinear among the primary figures, quadrilinear in the octahedron, and quinquelinear in the icosahedron. And so if one term of the ratio participates in the number 3, the ratio will be connected with the primary bodies; but if in the number 4, with the octahedron; and finally, if in the number 5, with the icosahedron. But in the feminine solids this kinship is more apparent, because the characteristic figure latent within follows upon the form of the angle: the tetragon in the octahedron, the pentagon in the icosahedron; and so 3 : 5 would go to the sectioned icosahedron for both reasons.

The second degree of kinship, which is genetic, is to be conceived as follows: First, some harmonic ratios of numbers are akin to one wedding or family, namely, perfect ratios to the single family of the cube; conversely, there is the ratio which is never fully expressed in numbers and cannot be demonstrated by numbers in any other way, except by a long series of numbers gradually approaching it: this ratio is called *divine*, when it is perfect, and it rules in various ways throughout the dodecahedral wedding. Accordingly, the following consonances begin to shadow forth that ratio: 1 : 2 and 2 : 3 and 2 : 3 and 5 : 8. For it exists most imperfectly in 1 : 2, more perfectly in 5 : 8, and still more perfectly if we add 5 and 8 to make 13 and take 8 as the numerator, if this ratio has not stopped being harmonic.

Further, in constructing the side of the figure, the diameter of the globe must be cut; and the octahedron demands its bisection, the cube and the tetrahedron its trisection, the dodecahedral wedding its quinquesection. Accordingly, the ratios between the figures are distributed according to the numbers which express those ratios. But the square on the diameter is cut too, or the square on the side of the figure is formed from a fixed part of the diameter. And then the squares on the sides are compared with the square on the diameter, and they constitute the following ratios : in the cube 1 : 3, in the tetrahedron 2 : 3, in the octahedron 1 : 2. Wherefore, if the two ratios are put together, the cubic and the tetrahedral will give 1 : 2; the cubic and the octahedral, 2 : 3; the octahedral and the tetrahedral, 3 : 4. The sides in the dodecahedral wedding are irrational.

Thirdly, the harmonic ratios follow in various ways upon the already constructed figures. For either the number of the sides of the plane is compared with the number of lines in the total figure; [274] and the following ratios arise : in the cube, 4 : 12 or 1 : 3; in the tetrahedron 3 : 6 or 1 : 2; in the octahedron 3 : 12 or 1 : 4; in the dodecahedron 5 : 30 or 1 : 6; in the icosahedron 3 : 30 or 1 : 10. Or else the number of sides of the plane is compared with the number of planes; then the cube

gives 4 : 6 or 2 : 3, the tetrahedron 3 : 4, the octahedron 3 : 8, the dodecahedron 5 : 12, the icosahedron 3 : 20. Or else the number of

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sides or angles of the plane is compared with the number of solid angles, and the cube gives 4 : 8 or 1 : 2, the tetrahedron 3 : 4, the octahedron 3 : 6 or 1 : 2, the dodecahedron with its consort 5 : 20 or 3 : 12 (*i.e.*, 1 : 4). Or else the number of planes is compared with the number of solid angles, and the cubic wedding gives 6 : 8 or 3 : 4, the tetrahedron the ratio of equality, the dodecahedral wedding 12 : 20 or 3 : 5. Or else the number of all the sides is compared with the number of the solid angles, and the cube gives 8 : 12 or 2 : 3, the tetrahedron 4 : 6 or 2 : 3, and the octahedron 6 : 12 or 1 : 2, the dodecahedron 20 : 30 or 2 : 3, the icosahedron 12 : 30 or 2 : 5.

Moreover, the bodies too are compared with one another, if the tetrahedron is stowed away in the cube, the octahedron in the tetrahedron and cube, by geometrical inscription. The tetrahedron is one third of the cube, the octahedron half of the tetrahedron, one sixth of the cube, just as the octahedron, which is inscribed in the globe, is one sixth of the cube which circumscribes the globe. The ratios of the remaining bodies are irrational.

The fourth species or degree of kinship is more proper to this work: the ratio of the spheres inscribed in the figures to the spheres circumscribing them is sought, and what harmonic ratios approximate them is calculated. For only in the tetrahedron is the diameter of the inscribed sphere rational, namely, one third of the circumscribed sphere. But in the cubic wedding the ratio, which is single there, is as lines which are rational only in square. For the diameter of the inscribed sphere is to the diameter of the circumscribed sphere as the square root of the ratio 1 : 3. And if you compare the ratios with one another, the ratio of the tetrahedral spheres is the square of the ratio of the cubic spheres. In the dodecahedral wedding there is again a single ratio, but an irrational one, slightly greater than 4 : 5. Therefore the ratio of the spheres of the cube and octahedron is approximated by the following consonances : 1 : 2, as proximately greater, and

3 : 5, as proximately smaller. But the ratio of the dodecahedral spheres is approximated by the consonances 4 : 5 and 5 : 6, as proximately smaller, and 3 : 4 and 5 : 8, as proximately greater.

But if for certain reasons 1 : 2 and 1 : 3 are arrogated to the cube, the ratio of the spheres of the cube will be to the ratio of the spheres of the tetrahedron as the consonances 1 : 2 and 1 : 3, which have been ascribed to the cube, are to 1 : 4 and 1 : 9, which are to be assigned to the tetrahedron, if this proportion is to be used. For these ratios, too, are as the squares of those consonances. And because 1 : 9 is not harmonic, 1 : 8 the proximate ratio takes its place in the tetrahedron. But by this proportion approximately 4 : 5 and 3 : 4 will go with the dodecahedral wedding.

For as the ratio of the spheres of the cube is approximately the cube of the ratio of the dodecahedral, so too the cubic consonances 1 : 2 and 2 : 3 are approximately the cubes of the consonances 4 : 5 and 3 : 4. For 4 : 5 cubed is 64 : 125, and 1 : 2 is 64 : 128. So 3 : 4 cubed is 27 : 64, and 1 : 3 is 27 : 81.

3. A SUMMARY OF ASTRONOMICAL DOCTRINE NECESSARY FOR SPECULATION INTO THE CELESTIAL HARMONIES

First of all, my readers should know that the ancient astronomical hypotheses of Ptolemy, in the fashion in which they have been unfolded in the *Theoricæ* of Peurbach and by the other writers of epitomes, are to be completely removed

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from this discussion and cast out of [275] the mind. For they do not convey the true lay out of the bodies of the world and the polity of the movements.

Although I cannot do otherwise than to put solely Copernicus' opinion concerning the world in the place of those hypotheses and, if that were possible, to persuade everyone of it; but because the thing is still new among the mass of the intelligentsia [*apud vulgus studiosorum*], and the doctrine that the Earth is one of the planets and moves among the stars around a motionless sun sounds very absurd to the ears of most of them: therefore those who are shocked by the unfamiliarity of this opinion should know that these harmonical speculations are possible even with the hypotheses of Tycho Brahe—because that author holds, in common with Copernicus, everything else which pertains to the lay out of the bodies and the tempering of the movements, and transfers solely the Copernican annual movement of the Earth to the whole system of planetary spheres and to the sun, which occupies the centre of that system, in the opinion of both authors. For after this transference of movement it is nevertheless true that in Brahe the Earth occupies at any time the same place that Copernicus gives it, if not in the very vast and measureless region of the fixed stars, at least in the system of the planetary world. And accordingly, just as he who draws a circle on paper makes the writing-foot of the compass revolve, while he who fastens the paper or tablet to a turning lathe draws the same circle on the revolving tablet with the foot of the compass or stylus motionless; so too, in the case of Copernicus the Earth, by the real movement of its body, measures out a circle revolving midway between the circle of Mars on the outside and that of Venus on the inside; but in the case of Tycho Brahe the whole planetary system (wherein among the rest the circles of Mars and Venus are found) revolves like a tablet on a lathe and applies to the motionless Earth, or to the stylus on the lathe, the midspace between the circles of Mars and Venus; and it comes about from this movement of the system that the Earth within it, although remaining motionless, marks out the same circle around the sun and midway

between Mars and Venus, which in Copernicus it marks out by the real movement of its body while the system is at rest. Therefore, since harmonic speculation considers the eccentric movements of the planets, as if seen from the sun, you may easily understand that if any observer were stationed on a sun as much in motion as you please, nevertheless for him the Earth, although at rest (as a concession to Brahe), would seem to describe the annual circle midway between the planets and in an intermediate length of time. Wherefore, if there is any man of such feeble wit that he cannot grasp the movement of the earth among the stars, nevertheless he can take pleasure in the most excellent spectacle of this most divine construction, if he applies to their image in the sun whatever he hears concerning the daily movements of the Earth in its eccentric—such an image as Tycho Brahe exhibits, with the Earth at rest.

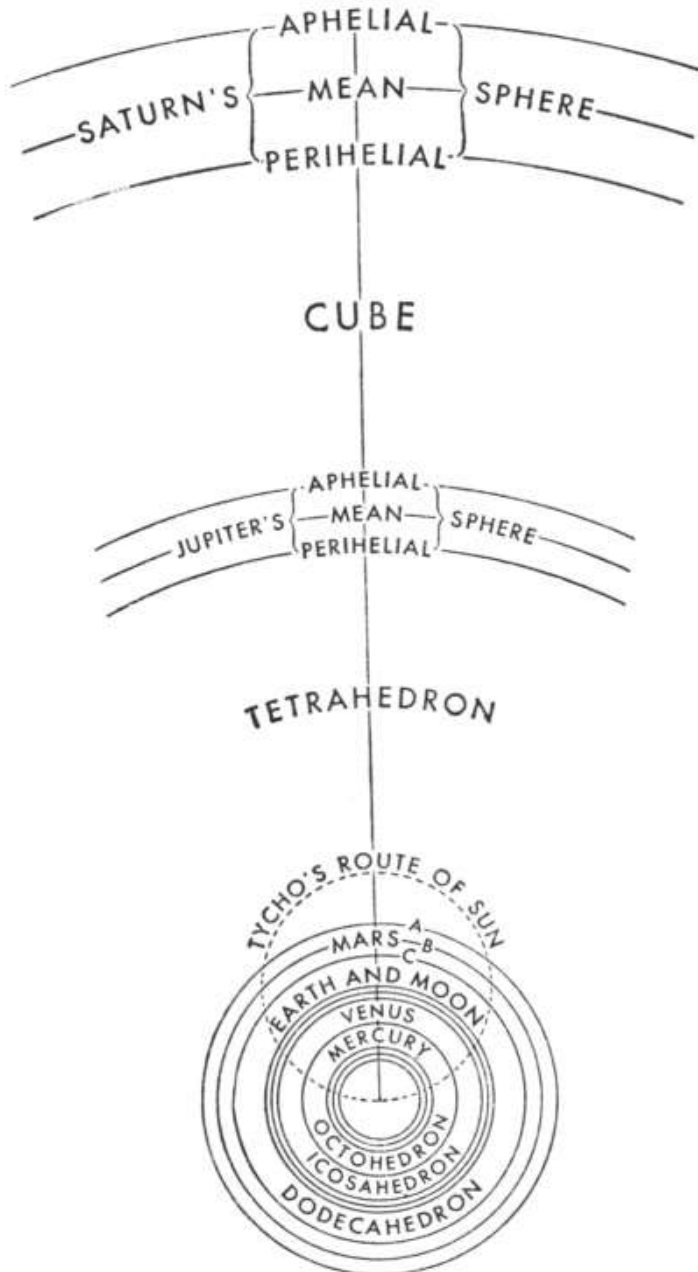
And nevertheless the followers of the true Samian philosophy have no just cause to be jealous of sharing this delightful speculation with such persons, because their joy will be in many ways more perfect, as due to the consummate perfection of speculation, if they have accepted the immobility of the sun and the movement of the earth.

Firstly [I], therefore, let my readers grasp that today it is absolutely certain among all astronomers that all the planets revolve around the sun, with the exception of the moon, which alone has the Earth as its centre: the magnitude

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of the moon's sphere or orbit is not great enough for it to be delineated in this diagram in a just ratio to the rest. Therefore, to the other five planets, a sixth, the Earth, is added, which traces a sixth circle around the sun, whether by its own proper movement with the sun at rest, or motionless itself and with the whole planetary system revolving.

Secondly [II]: It is also certain that all the planets are eccentric, *i.e.*, they change their distances from the



sun, in such fashion that in one part of their circle they become farthest away from the sun, [276] and in the opposite part they come nearest to the sun. In the accompanying diagram three circles apiece have been drawn for the single planets: none of them indicate the eccentric route of the planet itself; but the mean circle, such as *BE* in the case of Mars, is equal to the eccentric orbit, with respect to its longer diameter. But the orbit itself, such as *AD*, touches *AF*, the upper of the three, in one place *A*, and the lower circle *CD*, in the opposite place *D*. The circle *GH* made with dots and described through the centre of the sun indicates the route of the sun according to Tycho Brahe. And if the sun moves on this route, then absolutely all the points in this whole planetary system here depicted advance upon an equal route, each upon his own. And with one point of it (namely, the centre of the sun) stationed at one

point of its circle, as here at the lowest, absolutely each and every point of the system will be stationed at the lowest part of its circle. However, on account of the smallness of the space the three circles of Venus unite in one, contrary to my intention.

Thirdly [III]: Let the reader recall from my *Mysterium Cosmographicum*, which I published twenty-two years ago, that the number of the planets or circular routes around the sun was taken by the very wise Founder from the five regular solids, concerning which Euclid, so many ages ago, wrote his book which is called the *Elements* in that it is built up out of a series of propositions. But it has been made clear in the second book of this work that there cannot be more

regular bodies, *i.e.*, that regular plane figures cannot fit together in a solid more than five times.

Fourthly [IV]: As regards the ratio of the planetary orbits, the ratio between two neighbouring planetary orbits is always of such a magnitude that it is easily apparent that each and every one of them approaches the single ratio of the spheres of one of the five regular solids, namely, that of the sphere circumscribing to the sphere inscribed in the figure. Nevertheless it is not wholly equal, as I once dared to promise concerning the final perfection of astronomy. For, after completing the demonstration of the intervals from Brahe's observations, I discovered the following: if the angles of the cube [277] are applied to the inmost circle of Saturn, the centres of the planes are approximately tangent to the middle circle of Jupiter; and if the angles of the tetrahedron are placed against the inmost circle of Jupiter, the centres of the planes of the tetrahedron are approximately tangent to the outmost circle of Mars; thus if the angles of the octahedron are placed against any circle of Venus (for the total interval between the three has been very much reduced), the centres of the planes of the octahedron penetrate and descend deeply within the outmost circle of Mercury, but nonetheless do not reach as far as the middle circle of Mercury; and finally, closest of all to the ratios of the dodecahedral and icosahedral spheres—which ratios are equal to one another—are the ratios or intervals between the circles of Mars and the Earth, and the Earth and Venus; and those intervals are similarly equal, if we compute from the inmost circle of Mars to the middle circle of the Earth, but from the middle circle of the Earth to the middle circle of Venus. For the middle distance of the Earth is a mean proportional between the least distance of Mars and the middle distance of Venus. However, these two ratios between the planetary circles are still greater than the ratios of those two pairs of spheres in the figures, in such fashion that the centres of the dodecahedral planes are not tangent to the outmost circle of the Earth, and the centres of the icosahedral planes are not tangent to the outmost circle of Venus; nor, however, can this gap be filled by the semidiameter of the lunar sphere, by adding it, on the upper side, to the greatest distance of the Earth and subtracting it, on the lower, from the least distance of the same. But I find a certain other ratio of figures—namely, if I take the augmented dodecahedron, to which I have given the name of echinus, (as being fashioned from twelve quinquangular stars and thereby very close to the five regular solids), if I take it, I say, and place its twelve points in the inmost circle of Mars, then the sides of the pentagons, which are the bases of the single rays or points, touch the middle circle of Venus. In short: the cube and the octahedron, which are consorts, do not penetrate their planetary spheres at all; the dodecahedron and the icosahedron, which are consorts, do not wholly reach to theirs, the tetrahedron exactly touches both: in the first case there is falling short; in the second, excess; and in the third, equality, with respect to the planetary intervals.

Wherefore it is clear that the very ratios of the planetary intervals from the sun have not been taken from the regular solids alone. For the Creator, who is the very source of geometry and, as Plato wrote, "practices eternal geometry," does not stray from his own archetype. And indeed that very thing could be inferred from the fact that all the planets change their intervals throughout fixed periods of time, in such fashion that each has two marked intervals from the sun, a greatest and a least; and a fourfold comparison of the intervals from the

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sun is possible between two planets: the comparison can be made between either the greatest, or the least, or the contrary intervals most remote from one another, or the contrary intervals nearest together. In this way the comparisons made two by two between neighbouring planets are twenty in number, although on the contrary there are only five regular solids. But it is consonant that if the Creator had any concern for the ratio of the spheres in general, He would also have had concern for the ratio which exists between the varying intervals of the single planets specifically and that the concern is the same in both cases and the one is bound up with the other. If we ponder that, we will comprehend that for setting up the diameters and eccentricities conjointly, there is need of more principles, outside of the five regular solids.

Fifthly [V]: To arrive at the movements between which the consonances have been set up, once more I impress upon the reader that in the *Commentaries on Mars* I have demonstrated from the sure observations of Brahe that daily arcs, which are equal in one and the same eccentric circle, are not traversed with equal speed; but that these differing *delays in equal parts of the eccentric observe the ratio of their distances from the sun*, the source of movement; and conversely, that if equal times are assumed, namely, one natural day in both cases, the corresponding *true diurnal arcs* [278] *of one eccentric orbit have to one another the ratio which is the inverse of the ratio of the two distances from the sun*. Moreover, I demonstrated at the same time that *the planetary orbit is elliptical and the sun, the source of movement, is at one of the foci of this ellipse; and so, when the planet has completed a quarter of its total circuit from its aphelion, then it is exactly at its mean distance from the sun, midway between its greatest distance at the aphelion and its least at the perihelion*. But from these two axioms it results that *the diurnal mean movement of the planet in its eccentric is the same as the true diurnal arc of its eccentric at those moments wherein the planet is at the end of the quadrant of the eccentric measured from the aphelion, although that true quadrant appears still smaller than the just quadrant*. Furthermore, it follows that *the sum of any two true diurnal eccentric arcs, one of which is at the same [distance](#) from the aphelion that the other is from the perihelion, is equal to the sum of the two mean diurnal arcs*. And as a consequence, *since the ratio of circles is the same as that of the diameters, the ratio of one mean diurnal arc to the sum of all the mean and equal arcs in the total circuit is the same as the ratio of the mean diurnal arc to the sum of all the true*

eccentric arcs, which are the same in number but unequal to one another. And those things should first be known concerning the true diurnal arcs of the eccentric and the true movements, so that by means of them we may understand the movements which would be apparent if we were to suppose an eye at the sun.

Sixthly [VI]: But as regards the arcs which are apparent, as it were, from the sun, it is known even from the ancient astronomy that, among true movements which are equal to one another, that movement which is farther distant from the centre of the world (as being at the aphelion) will appear smaller to a beholder at that centre, but the movement which is nearer (as being at the perihelion) will similarly appear greater. Therefore, since moreover the true diurnal arcs at the near distance are still greater, on account of the faster movement, and still smaller at the distant aphelion, on account of the slowness of the movement, I demonstrated in the *Commentaries on Marsthat the ratio of the apparent diurnal arcs of one eccentric circle is fairly exactly the inverse ratio of the squares of their distances from the sun.* For example, if the planet one day when it is at

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a distance from the sun of 10 parts, in any measure whatsoever, but on the opposite day, when it is at the perihelion, of 9 similar parts: it is certain that from the sun its apparent progress at the aphelion will be to its apparent progress at the perihelion, as 81 : 100.

But that is true with these provisos: First, that the eccentric arcs should not be great, lest they partake of distinct distances which are very different—*i.e.*, lest the distances of their termini from the apsides cause a perceptible variation; second, that the eccentricity should not be very great, for the greater its eccentricity (*viz.*, the greater the arc becomes) the more the angle of its apparent movement increases beyond the measure of its approach to the sun, by Theorem 8 of Euclid's *Optics*; none the less in small arcs even a great distance is of no moment, as I have remarked in my *Optics*, Chapter 11. But there is another reason why I make that admonition. For the eccentric arcs around the mean anomalies are viewed obliquely from the centre of the sun. This obliquity subtracts from the magnitude of the apparent movement, since conversely the arcs around the apsides are presented directly to an eye stationed as it were at the sun. Therefore, when the eccentricity is very great, then the eccentricity takes away perceptibly from the ratio of the movements; if without any diminution we apply the mean diurnal movement to the mean distance, as if at the mean distance, it would appear to have the same magnitude which it does have—as will be apparent below in the case of Mercury. All these things are treated at greater length in Book V of the *Epitome of Copernican Astronomy*; but they have been mentioned here too because they have to do with the very terms of the celestial consonances, considered in themselves singly and separately.

Seventhly [VII]: If by chance anyone runs into those diurnal movements which are apparent [279] to those gazing not as it were from the sun but from the Earth, with which movements Book VI of the *Epitome of Copernican Astronomy* deals, he should know that their rationale is plainly not considered in this business. Nor should it be, since the Earth is not the source of the planetary movements, nor can it be, since with respect to deception of sight they degenerate not only into mere quiet or apparent stations but even into retrogradation, in which way a whole infinity of ratios is assigned to all the planets, simultaneously and equally. Therefore, in order that we may hold for certain what sort of ratios of their own are constituted by the single real eccentric orbits (although these too are still apparent, as it were to one looking from the sun, the source of movement), first we must remove from those movements of their own this image of the adventitious annual movement common to all five, whether it arises from the movement of the Earth itself, according to Copernicus, or from the annual movement of the total system, according to Tycho Brahe, and the winnowed movements proper to each planet are to be presented to sight.

Eighthly [viii]: So far we have dealt with the different delays or arcs of one and the same planet. Now we must also deal with the comparison of the movements of two planets. Here take note of the definitions of the terms which will be necessary for us. We give the name of *nearest apsides* of two planets to the perihelion of the upper and the aphelion of the lower, notwithstanding that they tend not towards the same region of the world but towards distinct and perhaps contrary regions. By *extreme movements* understand the slowest and the fastest of the whole planetary circuit; by *converging or converse extreme movements*, those which are at the nearest apsides of two planets—namely, at the

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perihelion of the upper planet and the aphelion of the lower; by *diverging or diverse*, those at the opposite apsides—namely, the aphelion of the upper and the perihelion of the lower. Therefore again, a certain part of my *Mysterium Cosmographicum*, which was suspended twenty-two years ago, because it was not yet clear, is to be completed and herein inserted. For after finding the true intervals of the spheres by the observations of Tycho Brahe and continuous labour and much time, at last, at last the right ratio of the periodic times to the spheres

*though it was late, looked to the unskilled man,
yet looked to him, and, after much time, came,*

and, if you want the exact time, was conceived mentally on the 8th of March in this year One Thousand Six Hundred and Eighteen but unfelicitously submitted to calculation and rejected as false, finally, summoned back on the 15th of May, with a fresh assault undertaken, outfought the darkness of my mind by the great proof afforded by my labor of seventeen years on Brahe's observations and meditation upon it uniting in one concord, in such fashion that I first believed I was dreaming

and was presupposing the object of my search among the principles. But it is absolutely certain and exact that *the ratio which exists between the periodic times of any two planets is precisely the ratio of the 3/2th power of the mean distances, i.e., of the spheres themselves*; provided, however, that the arithmetic mean between both diameters of the elliptic orbit be slightly less than the longer diameter. And so if any one take the period, say, of the Earth, which is one year, and the period of Saturn, which is thirty years, and extract the cube roots of this ratio and then square the ensuing ratio by squaring the cube roots, he will have as his numerical products the most just ratio of the distances of the Earth and Saturn from the sun. ¹ For the cube root of 1 is 1, and the square of it is 1; and the cube root of 30 is greater than 3, and therefore the square of it is greater than 9. And Saturn, at its mean distance from the sun, is slightly higher [280] than nine times the mean distance of the Earth from the sun. Further on, in Chapter 9, the use of this theorem will be necessary for the demonstration of the eccentricities.

Ninthly [IX]: If now you wish to measure with the same yardstick, so to speak, the true daily journeys of each planet through the ether, two ratios are to be compounded—the ratio of the true (not the apparent) diurnal arcs of the eccentric, and the ratio of the mean intervals of each planet from the sun (because that is the same as the ratio of the amplitude of the spheres), *i.e., the true diurnal arc of each planet is to be multiplied by the semidiameter of its sphere*: the products will be numbers fitted for investigating whether or not those journeys are in harmonic ratios.

Tenthly [X]: In order that you may truly know how great any one of these diurnal journeys appears to be to an eye stationed as it were at the sun, although this same thing can be got immediately from the astronomy, nevertheless it will also be manifest if you multiply the ratio of the journeys by the inverse ratio not of the mean, but of the true intervals which exist at any position on

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the eccentrics: *multiply the journey of the upper by the interval of the lower planet from the sun, and conversely multiply the journey of the lower by the interval of the upper from the sun.*

Eleventhly [XI]: And in the same way, if the apparent movements are given, at the aphelion of the one and at the perihelion of the other, or conversely or alternately, the ratios of the distances of the aphelion of the one to the perihelion of the other may be elicited. But where the mean movements must be known first, *viz., the inverse ratio of the periodic times, wherefrom the ratio of the spheres is elicited by Article VIII above*: then *if the mean proportional between the apparent movement of either one of its mean movement be taken, this mean proportional is to the semidiameter of its sphere (which is already known) as the mean movement is to the distance or interval sought*. Let the periodic times of two planets be 27 and 8. Therefore the ratio of the mean diurnal movement of the one to the other is 8 : 27.

Therefore the semidiameters of their spheres will be as 9 to 4. For the cube root of 27 is 3, that of 8 is 2, and the squares of these roots, 3 and 2, are 9 and 4. Now let the apparent aphelial movement of the one be 2 and the perihelial movement of the other $3\frac{1}{3}$. The mean proportionals between the mean movements 8 and 27 and these apparent ones will be 4 and 30. Therefore if the mean proportional 4 gives the mean distance of 9 to the planet, then the mean movement of 8 gives an aphelial distance 18, which corresponds to the apparent movement 2; and if the other mean proportional 30 gives the other planet a mean distance of 4, then its mean movement of 27 will give it a perihelial interval of $3\frac{3}{5}$. I say, therefore, that the aphelial distance of the former is to the perihelial distance of the latter as 18 to $3\frac{3}{5}$. Hence it is clear that if the consonances between the extreme movements of two planets are found and the periodic times are established for both, the extreme and the mean distances are necessarily given, wherefore also the eccentricities.

Twelfthly [XII]: It is also possible, from the different extreme movements of one and the same planet, to find the *mean movement*. The mean movement is not exactly the arithmetic mean between the extreme movements, nor exactly the geometric mean, but it is as much less than the geometric mean as the geometric mean is less than the [arithmetic] mean between both means. Let the two extreme movements be 8 and 10: the mean movement will be less than 9, and also less than the square root of 80 by half the difference between 9 and the square root of 80. In this way, if the aphelial movement is 20 and the perihelial 24, the mean movement will be less than 22, even less than the square root of 480 by half the difference between that root and 22. There is use for this theorem in what follows.

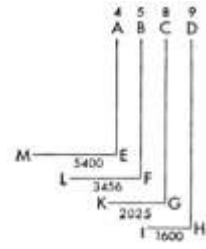
[281] Thirteenthly [XIII]: From the foregoing the following proposition is demonstrated, which is going to be very necessary for us: Just as the ratio of the mean movements of two planets is the inverse ratio of the $\frac{3}{2}$ th powers of the spheres, so the ratio of two apparent converging extreme movements always falls short of the ratio of the $\frac{3}{2}$ th powers of the intervals corresponding to those extreme movements; and in what ratio the product of the two ratios of the corresponding intervals to the two mean intervals or to the semidiameters of the two spheres falls short of the ratio of the square roots of the spheres, in that ratio does the ratio of the two extreme converging movements exceed the ratio of the corresponding intervals; but if that compound ratio were to exceed the

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ratio of the square roots of the spheres, then the ratio of the converging movements would be less than the ratio of their intervals. [1](#)

Let the ratio of the spheres be $DH : AE$; let the ratio of the mean movements be $HI : EM$, the $\frac{3}{2}$ th power of the inverse of the former.

Let the least interval of the sphere of the first be CG ; and the greatest interval of the sphere of the second be BF ; and first let $DH : CG$ comp. $BF : AE$ be smaller than the $\frac{1}{2}$ th power of $DH : AE$. And let GH be the apparent perihelial movement of the upper planet, and FL the aphelial of the lower, so that they are converging extreme movements.



I say that

$$GK : FL = BF : CG \\ BF^{3/2} : CG^{3/2}.$$

For

$$HI : GK = CG^2 : DH^2;$$

and

$$FL : EM = AE^2 : BF^2.$$

Hence

$$HI : GK \text{ comp. } FL : EM = CG^2 : DH^2 \text{ comp. } AE^2 : BF^2.$$

But

$$CG : DH \text{ comp. } AE : BF < AE^{1/2} : DH^{1/2}$$

by a fixed ratio of defect, as was assumed. Therefore too

$$HI : GK \text{ comp. } FL : EM = AE^{2/2} : DH^{2/2} \\ AE : DH$$

by a ratio of defect which is the square of the former. But by number VIII

$$HI : EM = AE^{3/2} : DH^{3/2}.$$

Therefore let the ratio which is smaller by the total square of the ratio of defect be divided into the ratio of the $\frac{3}{2}$ th powers; that is,

$$HI : EM \text{ comp. } GK : HI \text{ comp. } EM : FL = AE^{1/2} : DH^{1/2}$$

by the excess squared. But

$$HI : EM \text{ comp. } GK : HI \text{ comp. } EM : FL = GK : FL.$$

Therefore

$$GK : FL \quad AE^{1/2} : DH^{1/2}$$

by the excess squared. But

$$AE : DH = AE : BF \text{ comp. } BF : CG \text{ comp. } CG : DM;$$

And

$$CG : DH \text{ comp. } AE : BF \quad AE^{1/2} : DH^{1/2}$$

by the simple defect. Therefore

$$BF : CG \quad AE^{1/2} : DH^{1/2}$$

by the simple excess. But

$$GK : FL \quad AE^{1/2} : DT^{1/2}$$

but by the excess squared. But the excess squared is greater than the simple excess. Therefore the ratio of the movements GK to FL is greater than the ratio of the corresponding intervals BF to CG .

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In fully the same way, it is demonstrated even contrariwise that if the planets approach one another in G and F beyond the mean distances in H and E , in such fashion that the ratio of the mean distances $DH : AE$ becomes less than $DH^{1/2} : AE^{1/2}$, then the ratio of the movements $GK : FL$ becomes less than the ratio of the corresponding intervals $BF : CG$. For you need to do nothing more than to change the words *greater* to *less*, $>$ to $<$, *excess* to *defect*, and conversely.

In suitable numbers, because the square root of $4/9$ is $2/3$; and $5/8$ is even greater than $2/3$ by the ratio of excess $15/16$; and the square of the ratio $8 : 9$ [282] is the ratio $1600 : 2025$, *i.e.*, $64 : 81$; and the square of the ratio $4 : 5$ is the ratio $3456 : 5400$, *i.e.*, $16 : 25$; and finally the $3/2$ th power of the ratio $4 : 9$ is the ratio $1600 : 5400$, *i.e.*, $8 : 27$: therefore too the ratio $2025 : 3456$, *i.e.*, $75 : 128$, is even greater than $5 : 8$, *i.e.*, $75 : 120$, by the same ratio of excess (*i.e.*, $120 : 128$), $15 : 16$; whence $2025 : 3456$, the ratio of the converging movements, exceeds $5 : 8$, the inverse ratio of the corresponding intervals, by as much as $5 : 8$ exceeds $2 : 3$, the square root of the ratio of the spheres. Or, what amounts to the same thing, the ratio of the two converging intervals is a mean between the ratio of the square roots of the spheres and the inverse ratio of the corresponding movements.

Moreover, from this you may understand that the ratio of the diverging movements is much greater than the ratio of the $3/2$ th powers of the spheres, since the ratio of the $3/2$ th powers is compounded with the squares of the ratio of the aphelial interval to the mean interval, and that of the mean to the perihelial.

Footnotes

[1020:1](#) For in the *Commentaries on Mars*, chapter 48, page 232, I have proved that this Arithmetic mean is either the diameter of the circle which is equal in length to the elliptic orbit, or else is very slightly less.

[1022:1](#) Kepler always measures the magnitude of a ratio from the greater term to the smaller, rather than from the antecedent to the consequent, as we do today. For example, as Kepler speaks, $2 : 3$ is the same as $3 : 2$, and $3 : 4$ is greater than $7 : 8$.—C. G. Wallis.
